Universidad Simón Bolívar
Departamento de Matemáticas
Puras y Aplicadas
Enero - Marzo 2007

Nombre: ____________________

Carnet: ______________ Sección: _____

MA-3111 —Primer parcial (35 %) —

1. Halle las constantes $a, b, c \in \mathbb{R}$ tales que se cumple la igualdad

$$\operatorname{cotg} x \; \delta''_{\pi/4}(x) = a\,\delta_{\pi/4}(x) + b\,\delta'_{\pi/4}(x) + c\,\delta''_{\pi/4}(x)$$

Solución:

$$\Big(\operatorname{cotg} x \cdot \delta''_{\pi/4}(x)\,,\,\varphi(x)\Big) = \Big(\delta_{\pi/4}(x)\,,(\operatorname{cotg} x \cdot \varphi(x))''\Big)$$

$$(\operatorname{cotg})' = \frac{-1}{\operatorname{sen}^2 x}\,,(\operatorname{cotg} x)'' = \frac{2\operatorname{sen} x \cos x}{\operatorname{sen}^4 x} = \frac{2\cos x}{\operatorname{sen}^3 x}$$

$$\Rightarrow \quad (\operatorname{cotg} x \cdot \varphi(x))'' = \frac{2\cos x}{\operatorname{sen}^3 x}\cdot\varphi(x) - \frac{2}{\operatorname{sen}^2 x}\,\varphi'(x) + \operatorname{cotg} x \cdot \varphi''(x)$$

$$\Rightarrow \quad (\delta_{\pi/4}(x)\,,\,(\operatorname{cotg} x\cdot\varphi(x))'') = \frac{2\cos\pi/4}{\operatorname{sen}^3\pi/4}\,\varphi\left(\frac{\pi}{4}\right) - \frac{2}{\operatorname{sen}^2\frac{\pi}{4}}\,\varphi'\left(\frac{\pi}{4}\right) + \operatorname{cotg}\frac{\pi}{4}\cdot\varphi''\left(\frac{\pi}{4}\right)$$

$$= \quad 4\cdot\varphi\left(\frac{\pi}{4}\right) - 4\varphi'\left(\frac{\pi}{4}\right) + \varphi''\left(\frac{\pi}{4}\right) = \big(4\delta_{\pi/4}(x)\,,\,\varphi(x)\big) + \Big(4\delta'_{\pi/4}(x)\,,\,\varphi(x)\Big) + \Big(\delta''_{\pi/4}(x)\,,\,\varphi(x)\Big)$$

$$= \quad \Big(\delta_{\pi/4}(x) + 4\delta'_{\pi/4}(x) + \delta''_{\pi/4}(x)\,,\,\varphi(x)\Big)\;,\;\forall\,\varphi\in\mathcal{D}$$

$$\Rightarrow \quad \operatorname{cotg} x\cdot\delta''_{\pi/4}(x) = 4\cdot\delta_{\pi/4}(x) + 4\delta'_{\pi/4}(x) + \delta''_{\pi/4}(x)$$

$$\Rightarrow \quad a = 4\,,\; b = 4\,,\; c = 1$$

2. Para $n = 1, 2, 3, \ldots$ halle el valor de $\Im_n$ si

$$\Im_n := \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \text{ donde } f(x) = \begin{cases} 0 & , \quad x \notin [0, 2\pi] \\ 3 & , \quad x \in [0, \pi] \\ 6 - \frac{3x}{\pi} & , \quad x \in [\pi, 2\pi] \end{cases}$$

Solución:

$$
\begin{aligned}
& f_g^{'}(x) = 3\delta(x) + f_c^{'}(x) \\
& f_g^{''}(x) = 3\delta^{'}(x) + \frac{-3}{\pi}\,\delta_{\pi}(x) + \frac{3}{\pi}\,\delta_{2\pi}(x) \\
& \frac{1}{\pi}\left(f_g^{''}(x)\,,\,\cos nx\right) = \frac{1}{\pi}\left(f(x)\,,\,(\cos nx)^{''}\right) = -n^2\cdot\frac{1}{\pi}(f(x)\,,\,\cos nx) = -n^2\,\Im_n \\
\Rightarrow\ & \Im_n = \frac{-1}{\pi n^2}\left(f_g^{''}(x)\,,\,\cos nx\right) \Rightarrow \Im_n = \frac{-1}{\pi n^2}\left(3\delta^{'}(x) - \frac{3}{\pi}\,\delta_{\pi}(x) + \frac{3}{\pi}\,\delta_{2\pi}(x)\,,\,\cos nx\right) \\
= \ & \frac{3}{\pi^2 n^2}\left(\delta(x)\,,\,(\cos nx)^{'}\right) + \frac{3}{\pi^2 n^2}\left(\delta_{\pi}(x) - \delta_{2\pi}(x)\,,\,\cos nx\right) \\
= \ & \frac{-3n}{\pi n^2}(\delta(x)\,,\,\operatorname{sen} nx) + \frac{3}{\pi^2 n^2}(\cos(n\pi) - \cos(n\cdot 2\pi)) = \frac{-3}{\pi n}\operatorname{sen} 0 + \frac{3}{\pi^2 n^2}\left((-1)^n - 1\right) \\
= \ & \frac{3}{\pi^2 n}\left((-1)^n - 1\right). \Rightarrow \Im_n = \frac{3}{\pi^2 n^2}\left((-1)^n - 1\right).
\end{aligned}
$$

3. Halle $u(t)$ si $u(t) = (3t-2)h(t) * h(t)\operatorname{sen} t$
   Solución:

$$
\begin{aligned}
u(t) &= 3t\cdot h(t) * h(t)\operatorname{sen} t - 2h(t) * h(t)\operatorname{sen} t \\
U(z) &\doteq u(t) \Rightarrow U(z) = 3\cdot\frac{1}{z^2}\cdot\frac{1}{z^2+1} - 2\cdot\frac{1}{z}\cdot\frac{1}{z^2+1}; \\
& \frac{1}{z^2}\cdot\frac{1}{z^2+1} = \frac{a}{z^2} + \frac{b}{z^2+1} = \frac{1}{z^2} - \frac{1}{z^2+1}\,,\ \frac{1}{z}\cdot\frac{1}{z^2+1} = \frac{a}{z} + \frac{bz+c}{z^2+1} = \frac{z}{z^2+1}. \\
\Rightarrow\ & U(z) = 3\left(\frac{1}{z^2} - \frac{1}{z^2+1}\right) - 2\left(\frac{1}{z} - \frac{z}{z^2+1}\right) = 3\cdot\frac{1}{z^2} - 2\frac{1}{z} - 3\cdot\frac{1}{z^2+1} + 2\cdot\frac{z}{z^2+1} \\
\Rightarrow\ & u(t) = 3th(t) - 2h(t) - 3h(t)\operatorname{sen} t + 2h(t)\cos t \\
\Rightarrow\ & u(t) = h(t)\cdot(3t - 2 - 3\operatorname{sen} t + 2\cos t).
\end{aligned}
$$

4. Halle la solución del problema inicial

$$
\begin{cases} \mathcal{L}v(t) = -3t \\ v(0) = 0 \quad , \quad v^{'}(0) = 5 \end{cases}, \text{ donde } \mathcal{L} = \frac{d^2}{dt^2} - 2\frac{d}{dt} + 1.
$$

Solución:

$$
\begin{aligned}
\mathcal{L}v &= v^{''} - 2v^{'} + 1. \\
]\ u(t): &= h(t)v(t) \Rightarrow u^{'}(t) = \delta(t)\cdot v(t) + h(t)v^{'}(t) = \delta(t)\cdot v(0) + h(t)v^{'}(t) = h(t)\cdot v^{'}(t)\,, \\
&\Rightarrow u^{''}(t) = \Big(h(t)v^{'}(t)\Big)^{'} = \delta(t)v^{'}(t) + h(t)v^{''}(t) = \delta(t)\cdot v^{'}(0) + h(t)v^{''}(t) \\
&= \delta(t)\cdot 5 + h(t)v^{''}(t). \Rightarrow \mathcal{L}u = u^{''} - 2u^{'} + u = 5\cdot\delta(t) + h(t)v^{''}(t) - 2h(t)v^{'}(t) + h(t)v(t) \\
&= 5\cdot\delta(t) + h(t)\mathcal{L}v = 5\cdot\delta(t) + h(t)\cdot(-3t) \Rightarrow u^{''} - 2u^{'} + u = 5\cdot\delta(t) - 3t\cdot h(t). \\
&\quad]\ U(z) = u(t) \Rightarrow (z^2 - 2z + 1)U(z) = 5 - 3\cdot\frac{1}{z^2} \Rightarrow U(z) = \frac{5}{(z-1)^2} - 3\cdot\frac{1}{z^2}\cdot\frac{1}{(z-1)^2}\,; \\
&\quad \frac{1}{z^2}\cdot\frac{1}{(z-1)^2} = \frac{a}{z^2} + \frac{b}{z} + \frac{c}{(z-1)^2} + \frac{d}{(z-1)} \Rightarrow 1 = a(z-1)^2 + bz\cdot(z-1)^2 + c\cdot z^2 \\
&\quad +d\cdot z^2(z-1) \Rightarrow a = 1,\ b = 2,\ c = 1,\ d = -2 \Rightarrow \\
&\quad U(z) = \frac{5}{(z-1)^2} - 3\left(\frac{1}{z^2} + \frac{2}{z} + \frac{1}{(z-1)^2} + \frac{-2}{(z-1)}\right) \\
&= 2\cdot\frac{1}{(z-1)^2} + 6\cdot\frac{1}{z-1} - 3\cdot\frac{1}{z^2} - 6\cdot\frac{1}{z} \\
&\Rightarrow u(t) = 2e^t\cdot th(t) + 6e^t h(t) - 3th(t) - 6h(t) \\
&\Rightarrow u(t) = h(t)\cdot(2e^t + 6e^t - 3t - 6) \equiv h(t)\cdot v(t) \\
&\Rightarrow v(t) = 2te^t + 6e^t - 3t - 6.
\end{aligned}
$$